CHAPTER 1

A cartesian introduction

1 Proofs, applications, and other mathematical activities

Why *is* there a whole field of inquiry, a discipline if you like, called the philosophy of mathematics? This unusual question, the very title of this book, will not begin to be examined with care until Chapter 3, but two summary answers can be stated at once.

First, because of the *experience* of some demonstrative proofs, the experience of proving to one's complete satisfaction some new and often unlikely fact. Or simply experiencing the power and conviction conveyed by a good proof that one is taught, that one reads, or has explained to one. How can mere words, mere ideas, sometimes mere pictures, have those effects?

Second, because of the richness of applications of mathematics, often derived by thinking at a desk and toying with a pencil. Or more poetically, in the words of the historian of science A. C. Crombie (1994 1, ix), 'the enigmatic matching of nature with mathematics and of mathematics by nature'.

Thus this book is a series of philosophical thoughts about proofs, applications, and other mathematical activities.

In line with the authors of my second and third epigraphs, Lakatos and Stein, I think of proving and using mathematics as activities, not as static done deeds. But at once a word of caution. One does not need proofs to think mathematically: it is a contingent historical fact that proof is the present (and was once the Euclidean) gold standard of mathematicians. Our common distinction between pure mathematics and applications is likewise by no means inevitable. (These assertions to be established in Chapters 4 and 5.) Thus the philosophical difficulties prompted by proofs and applications have arisen because of the historical trajectory of mathematical practice, and are in no sense 'essential' to the subject. What are often presented as very clear ideas, namely proof and application, turn out to be much more fluid than one might have imagined.

Two figures haunt the philosophy of mathematics: Plato and Kant. Plato, as I read him, was bowled over by the experience of demonstrative proof. Kant, as I understand him, crafted a major part of his philosophy to account for the enigmatic ability of mathematics, seemingly the product of human reason, so perfectly to describe the natural world. Thus Plato inaugurated philosophizing about mathematics, and Kant created a whole new problematic. Another version of my answer to the question, 'Why philosophy of mathematics?' is, therefore, 'Plato and Kant'.

A third figure haunts my own philosophical thinking about mathematics: Wittgenstein. I bought my copy of the *Remarks on the Foundation of Mathematics* on 6 April 1959, and have been infatuated ever since. What follows in this book is not what is called 'Wittgensteinian', but he hovers in the background. Hence it is well to begin with a warning.

2 On jargon

My first epigraph, 'For mathematics is after all an anthropological phenomenon', is taken out of context from a much longer paragraph, some of which is quoted in Chapter 2, §13 (henceforth, §2.13). The long paragraph is itself part of an internal dialogue stretching over a couple of pages. The words I used for the epigraph sound good, but bear in mind the last recorded sentence of Wittgenstein's 1939 *Lectures on the Foundations of Mathematics* (1976: 293): 'The seed I'm most likely to sow is a certain jargon.'

A good prediction! Recall some jargon derived from his words:

Language game
Form of life
Family resemblance
Rule following considerations
Surveyable/perspicuous
Hardness of the logical 'must'
etc. . . . and:
Anthropological phenomenon
What we are supplying are remarks on the natural history of mankind.

Not to mention things he did not exactly say – 'Don't ask for the meaning, ask for the use.' (For what he did say, see §6.23.)

These are wonderful phrases. But they should be treated with caution and read in context. They all too easily invite the feeling of understanding. They are often cited as if they were at the end, not in the middle, of a series of thoughts. This book will quote scraps from Wittgenstein quite often. Hence it is prudent to begin with this *caveat emptor*, lest I forget.

Another notice to start with. I shall often allude to one or another of Wittgenstein's observations. But I shall not mention his thoughts about infinity, intuitionism, Cantor, or Gödel. And I will not discuss his ideas about following a rule, for a reason to be stated in §3.24.

3 Descartes

Descartes also created a certain jargon. These two exceptional philosophers may have far more in common than is usually noticed (Hacking 1982). Be that as it may, what follows began with adolescent infatuation with Wittgenstein, and its present form began as the René Descartes Lectures at the University of Tilburg.

The title of this biennial series of lectures is honorific, recalling only that Descartes (1596–1650) chose to live and work in the free intellectual climate of the Netherlands. There, between 1628 and 1649, he created a philosophy that would dominate the modern age. I was under no obligation to pay homage to the historical Descartes, but two distinct Cartesian themes will recur in what follows.

The first is the application of arithmetic to geometry – a central example of the application of mathematics to mathematics, itself a case of the application of mathematics to the sciences (including mathematics). This step in the great chain of applications, namely the application of arithmetic to geometry, is of particular note.

The second theme is the extraordinarily different conceptions of proof that seem to have been held by Descartes on the one hand, and, a little later, by Leibniz on the other. This is an extreme case of the great variety in types of proof that are used by mathematicians. This first modern confrontation between two ideal visions of proof foreshadows topics much discussed today, namely the impact of fast computation on the practice of 'pure' mathematics. Perhaps more important is a question that has newly become vital, when not long ago we thought we knew the answer: what *is* a mathematical proof?

Both topics are examples of later developments in this book. Neither theme is commonly thought of as Cartesian. Neither plays much of a role in contemporary philosophizing about mathematics, or in Cartesian scholarship. Thus this first 'Cartesian' chapter launches us into relatively uncharted territory, which will later be discussed in parallel with the well-mapped regions that today's philosopher expects to enter.

If there is any 'methodological' idea in this book, it is this: in philosophical thinking about mathematics, it is good to start by grasping straws from the tangled nest that is mathematics, straws that have not been much picked over recently. In the course of this chapter I shall also mention a number of other topics that will come up later, but which have received very little, if any, attention from philosophers of mathematics, although they certainly interest mathematicians. So this chapter comes, not in lieu of an introduction, but as a working introduction.

Since the two themes, of proof and application, appear to be unrelated, this chapter is divided into two parts. I say *appear*, for Wittgenstein may have thought differently: 'I should like to say: mathematics is a MOTLEY of techniques of proof. – And upon this is based its manifold applicability and its importance' (Wittgenstein 1978 (henceforth *RFM*): III, §46, 176). As already noted in my Foreword, the two most common substantive nouns (and their cognates) in the first edition of the *Remarks* (1956) were 'proof' (*Beweis*) and 'application' (*Anwendung*). But beware of jargon!

A APPLICATION

4 Arithmetic applied to geometry

> I have a vivid and happy memory of my first reading of Descartes, for it was with unbounded enthusiasm that I devoured the *Discourse on Method*, sitting in the shade of a tree in the Borghese Gardens on Rome in the summer of 1970. (Gaukroger 1995: vii)

People have been having this sort of experience since 1637, when the *Discourse* was published. Its full title is the *Discourse on the Method of Properly Conducting One's Reason and Seeking the Truth in the Sciences*. We tend to think of it as a free-standing work, and forget that it was published as the preface to three scientific treatises, the *Meteors*, the *Dioptrics*, and the *Geometry*.

The *Meteors* is about 'meteorology', understood as ranging from vapours to the rainbow. It is a perfect example of what T. S. Kuhn called a pre-paradigmatic discussion. Later he withdrew the idea of science before paradigms, but he was still pointing at something important in the study of some group of phenomena (Kuhn 1977: 295, n. 4). Kuhn's own example was *heat* at the time of Francis Bacon. In the case of Bacon, the heat from rotting manure is listed alongside the heat of the sun. Likewise in the *Meteors*, a vast range of phenomena are grouped together with no clear idea (for the modern reader) of what on earth they have to do with each other, except that they are above the earth but not, apparently, in the heavens.

The *Dioptrics*, about the theory of light, is in a much more advanced state, for much of it is about geometrical optics, followed by the theory of reflection and refraction, to which Descartes made important contributions, not always correct. It concludes with a physico-physiological discussion of how the eye works.

The *Geometry* is a thoroughly modern work, a surprising amount of which can be read by an interested reader without much difficulty today. I like to think of this third essay as one model of how to conduct one's reason and seek truth in a science.

I am not about to embark on an exposition of Cartesian philosophy, neither that of 'Mr Cogito' or 'Mr Dualist' on the one hand, nor, on the other hand, Descartes the mechanical philosopher, so brilliantly described by Martial Guéroult (1953) and then Daniel Garber (1992). What follows is one very thin slice from a very rich cake.

5 Descartes' *Geometry*

The *Geometry* made plain, for all the world to see, that there are profound relations between geometry and arithmetic. We now find it more natural to say that Descartes applied algebra to geometry, but Descartes did not see things in quite that way. He began by comparing arithmetical operations to geometrical constructions, and after listing many bits of arithmetical terminology, said that he would 'not hesitate to introduce these arithmetical terms into geometry for the sake of greater clearness' (Descartes 1954: 5).

Connections between geometry and arithmetic did not originate with Descartes. Many before him grappled with the same issues. The usual suspects are, in France, François Viète (1540–1603), and in Germany, the Jesuit astronomer and mathematician Christopher Clavius (1538–1612). In Italy we might go less to the university mathematicians than to the abacus tradition: that is, the work of men of commerce who actually used arithmetic, rather than the great innovators such as Cardano and Tartaglia who solved cubic equations and the like.

The eminent historian of science, John Heilbron, included in his *Galileo* (2010) a striking 'rational reconstruction' of Galileo's thought. It is an amazing dialogue in which 'Galileo' tries not only to put algebra together with geometry, but also to express time in such a way that it enters into geometrical/algebraic presentations. Minkowski portended?

Probably the history of arithmetic applied to geometry will go back further than the Italians, to the House of Wisdom in Baghdad, where algebra as we know it matured in the tenth century, and probably to Pappus of Alexandria (about 290–350). Descartes did not come from nowhere but from everywhere. He owed far more of his geometrical results, about, for example, conic sections, to Apollonius (262–190 BCE) than he cared to acknowledge. Yet it was he who turned all these explorations into a perspicuous whole that novices could master.

6 An astonishing identity

Many difficult problems in geometry can be solved by turning them into arithmetic and algebra, and many problems in the theory of numbers and algebra can be solved by turning them into geometry. This continues, as we shall soon illustrate, from the time of Descartes to the present day. It is as if geometry and the theory of numbers turn out to be about the same stuff. I find this astonishing.

In contrast most people take it for granted if they think about it at all. One scholar whom I greatly respect, and to whom I expressed my concern, said, 'it is obviously over-determined'. Maybe. But let us begin modestly, thinking only in terms of the application of one science to another, and in particular of Descartes 'applying' arithmetic and algebra to plane and later solid geometry.

7 Unreasonable effectiveness

Algebra, born of arithmetic, can be applied to geometry. That is the first indubitable example of the unreasonable effectiveness of mathematics developed for one purpose, applied to mathematics developed for another purpose. (Others, more knowledgeable than I, might cite Archimedes over Descartes.)

My tag, 'unreasonable effectiveness', is a deliberate play on the title of a famous essay by Eugene Wigner: 'The unreasonable effectiveness of mathematics in the natural sciences' (1960). Wigner's phrase has become a cliché. He was surprised that quite often 'pure mathematics' developed, as Wigner put it, primarily for aesthetic purposes, has turned out to be an invaluable tool in the sciences. The most familiar example – perhaps suspiciously overused – is the way in which Riemann's non-Euclidean geometry was precisely the tool needed for special relativity.

Wigner's question has been asked often, although seldom with the persistence of Mark Steiner's *The Applicability of Mathematics as a Philosophical Problem* (1998). We shall turn to these issues later. Descartes reminds us that there is another question which is seldom addressed. How come mathematics developed in one domain, with one set of interests, turns out to make a critical difference to mathematics developed in another domain, with an apparently different subject matter? My play on Wigner's 'unreasonable effectiveness' of mathematics, applied not to physics but to mathematics, has already been made by Corfield (2003).

Why should the application of algebra to geometry be 'unreasonable', or at least unexpected? Here are three reasons to start with.

According to the traditional history of 'Western' mathematics, geometry is Greek, and arithmetic and algebra start in India and pass through Persia and Islam. On that account, they have distinct historical origins. Second, Kant plausibly made arithmetic out to be the synthetic a priori truths of time, while geometry constitutes the synthetic a priori truths of space. Third, much contemporary cognitive science locates the 'number sense' in parts of the brain distinct from those deployed in spatial sensibility.

It has become something of a truism that people in most societies tend to imagine the series of numbers as arranged on a line. Remember, however, most truisms, if true at all, are at best 'true for the most part'. Núñez (2011) argues that this linear representation of the numbers is recent even in the West, and that there is no trace of it in Babylonian mathematics or among many Amazonian peoples. A more interesting thought is about Mayan civilization. The world did not come to an end at the end of 2012, as some ludicrous misreadings of Mayan calendars led the fanciful to fear, but the Maya may well have had a circular sense of number rather than a linear one. But of course that is only another geometrical, spatial, representation of number.

The interconnections between geometry and algebra are, nevertheless, almost too good to be true. Geometry is spatial, and algebra is a child of arithmetic. Arithmetic is for counting, a process that, as Kant emphasized, takes place in time. I don't mean that you cannot do arithmetic without time, or know cardinalities without counting. You can usually tell just by looking that a group has exactly four or five or maybe even six members. (That's called 'subitizing': infants can discriminate between groups of two and three. So can monkeys, pigeons, crows, and dolphins.) After that, you have to count to be sure. Counting, Kant would have insisted, progresses in time, and counting is almost certainly not innate but had to be learned. But you can often tell, just by looking, that one group of things is larger than another. You can tell that two sets have the same number of members by matching them, like Frege's waiter (Frege 1952: 81). Nevertheless, I do think that Kant's initial location of arithmetic in time is important, and brings out a difference between it and geometry in space.

Although constructing a figure or even drawing a line takes time, Kant was right to say you can see a circle is a circle just by looking, and even if you can't name an irregular shape, you can see it is a shape. Space and time, manifested in geometry and arithmetic, are the twin poles of Kant's Transcendental Aesthetic. (Why 'Aesthetic'? Because the aesthetic deals with what he called sensibility, not reason.) He used space and time, geometry and arithmetic, as the road to an entire philosophy that has haunted us ever since.

All right, it will be protested, Kant did think that arithmetic and geometry are fundamentally different, but that part of his philosophy has not stood up very well. Most would say that his ideas about geometry were refuted by special relativity in 1905; his views about space and time were refuted by general relativity in 1916; his views about causation were refuted by the second wave of quantum mechanics in 1926. Early in the twentieth century L. E. J. Brouwer (1881–1966) did found Intuitionism on Kantian principles about continuing sequences in time, but his chief legacy seems to be constructive mathematics, which seldom makes use of Kant's motivating thought. So why should we, today, be impressed by the well-known fact that arithmetic can be applied to geometry?

One answer, as already suggested, derives from contemporary cognitive science. There is the now popular modular view of the brain and cognition. It is a plausible conjecture that the modules that enable us to navigate our spatial environment, and invite geometry, are distinct from those that give us *The Number Sense*, to use Stanislas Dehaene's title (1997). Figuratively speaking, we have two distinct cognitive systems, each run by its own neural network in the brain. Very plausible. (Just the sort of thing that Kant might have expected!) There is at least something of a consensus, at present, that there are distinct 'core systems', one group of which enables judgements of numerosity, and another of which enables simple geometrical judgements. Yet somehow the two, arithmetic and geometry, turn out to be about the same stuff.

8 The application of geometry to arithmetic

Descartes applied arithmetic to geometry. The tables can be turned, and geometry applied to numbers. Let us take, for example, a name already mentioned, that of Hermann Minkowski (1864–1909). He was one of Einstein's teachers, and the better mathematician. It was Minkowski who, in 1907, realized that the special theory of relativity should be conceptualized in four-dimensional space-time. In 1908 he gave a fundamental lecture to the annual meeting of German scientists and physicians; it began with the rousing words:

> Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality. (Minkowski 1909)

Kant on space and time demolished in a sentence? Maybe. But Minkowski is relevant here for quite a different reason. He was first famed as an extraordinary innovator in number theory. Yet he made most of his discoveries in number theory – and then in mathematical physics and in the theory of relativity – by using geometry. (See Galison 1979, a fascinating paper he wrote as an undergraduate.) That is the inverse of the Cartesian application of algebra to geometry.

9 The application of mathematics to mathematics

The application of geometry to number theory, and of algebra to geometry, are special cases of the application of mathematics to mathematics. Because of my concern with the proposed distinct cognitive origins of geometry and numbers, I find it deeply perplexing. The more general topic of the application of one bit of mathematics to another bit of mathematics has received curiously little attention from philosophers. It is a constant source of both delight and achievement among mathematicians. Many of the examples of recent mathematics that I shall use in what follows take an idea from one branch of mathematics and use it to create or to solve a fundamental problem in another.

Kenneth Manders is the rare philosopher who has taken the topic very seriously. 'We are accustomed to the idea that something *prima facie* intellectually powerful is happening in empirical applications of mathematics, but philosophers . . . have yet to admit a parallel in the math-math world.' I quote from 'Why apply math?', an unpublished paper written in the 1990s. His title is at first glance misleading, for it suggests an interest in empirical applications, but which were precisely not his target. Yet the title is not inapt, for he thinks that grounds for the applicability of mathematics may 'lie in features shared by math-empirical and math-math applications alike'. I share his suspicion.

Manders explains philosophical indifference to 'math-math' by the twentieth-century attitude that mathematics is all deduction, all proofs. 'From this point of view, there is indeed nothing special about applying a mathematical theorem in a mathematical proof: it's just proofs and more proofs.' As illustrations of this attitude he cites a letter of Hilbert's to Frege (21 February 1899; Frege 1980) and a famous paper by C. G. Hempel, first published in 1945 in the *American Mathematical Monthly*. It is clear that Hempel believed that the application of mathematics to any field was a matter of interpreting a set of axioms in a new context: what Manders calls the 'interpretive' conception of the application of mathematics. I am tempted to go further: after Gödel and Tarski, logicians and philosophers were justifiably obsessed with a semantic approach to mathematics. That deflected attention from what is being done when one domain of mathematics is applied to another. To paraphrase Manders, it made it look like models and more models.

The idea identified by Manders, and which he says has misled us, and which I connect with the semantic approach, is of an abstract formal system within which one makes valid deductions; then there are various interpretations of the system. A body of mathematics is applied to a subject matter when the subject matter provides an interpretation for the mathematics. All deductions valid in the one lead, under the interpretation, to sound conclusions in the other. And that's all there is to applying one branch of mathematics to another. Manders observes that things are not like that at all.

He takes as his example of 'math-math' application not Descartes, but a theorem discovered by Girard Desargues (1591–1661) in 1647, ten years after Descartes had published his *Geometry*. The result lay dormant for a long time, until the revival of projective geometry in the nineteenth century. Manders argues that the resulting reconception of Euclidean geometry is not just a matter of reinterpretation of axioms, but a change in the ways in which space can be conceived and presented to the mind. Projective geometry (to simplify Manders' subtle paper) was applied to Euclidean geometry, and our understanding of both was affected by the interchange. (A little more about projective geometry in §§4.5 and 5.11.) The arid one-way tale of axioms and interpretations completely ignores the dynamics of the situation, or so Manders argues. This is an important insight, which is becoming increasingly widely shared. It will be illustrated in detail in connection with the 'pure' and 'applied' mathematics of rigidity in §§5.24–26.

The theme, of how technical inventions can alter human possibilities for thinking about space, is itself worthy of philosophical thought. That is what Hermann Minkowski did with his diagrams, and, in the 1960s, what Roger Penrose and Brandon Carter did with conformal diagrams, now often called Penrose diagrams. With a lot of training, a very clever young person can learn to represent the causal properties of infinite dimensional space on a flat piece of paper or a whiteboard. And then to think productively about black